Appointment Scheduling with Limited Distributional Information

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1. Online Supplement

Proof of Theorem 4

By using Proposition 2, and by denoting $\alpha_i = \rho_{i1}$, we have (12) is equivalent to

$$\begin{aligned} \min_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{s}} \quad & \textstyle\sum_{i=1}^{n} (\lambda_i + \mu_i \alpha_i) \\ \text{s.t.} \quad & \textstyle\sum_{i=k}^{\min\{n,j\}} \max_{p_i \in D_i} ((p_i - s_i)\pi_{ij} - \alpha_i p_i - \lambda_i) \le 0 \qquad \text{for } 1 \le k \le n, 1 \le k \le j \le n+1 \\ & \boldsymbol{s} \in \mathcal{S}. \end{aligned} \tag{A.1}$$

For $1 \le i \le n$ and $1 \le i \le j \le n+1$, we have

$$\max_{p_i \in D_i} (p_i - s_i)\pi_{ij} - \alpha_i p_i - \lambda_i = \begin{cases} (\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i) - \pi_{ij} s_i - \lambda_i & \text{if } \pi_{ij} \ge \alpha_i \\ (\pi_{ij} - \alpha_i)(\mu_i - \underline{d}_i) - \pi_{ij} s_i - \lambda_i & \text{if } \pi_{ij} < \alpha_i. \end{cases}$$

Then (A.1) is equivalent to

$$\sum_{i=k}^{\min\{n,j\}} \pi_{ij} s_i + \lambda_i \ge \sum_{i=k}^{\min\{n,j\}} \max\big((\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i), (\pi_{ij} - \alpha_i)(\mu_i - \underline{d}_i)\big) \quad \text{for } 1 \le k \le n, 1 \le k \le j \le n+1. \tag{A.2}$$

By introducing new variables ξ to replace $\max\big((\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i), (\pi_{ij} - \alpha_i)(\mu_i - \underline{d}_i)\big)$, (A.2) is equivalent to

$$\begin{aligned} & \textstyle\sum_{i=k}^{\min\{n,j\}} \xi_{ij} \le \sum_{i=k}^{\min\{n,j\}} \lambda_i + s_i \pi_{ij} && \text{for } 1 \le k \le n, 1 \le k \le j \le n+1 \\ & \xi_{ij} \ge (\pi_{ij} - \alpha_i)(\mu_i + \bar{d}_i) && \text{for } 1 \le i \le n, 1 \le i \le j \le n+1 \\ & \xi_{ij} \ge (\pi_{ij} - \alpha_i)(\mu_i - \underline{d}_i) && \text{for } 1 \le i \le n, 1 \le i \le j \le n+1. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5

In this proof, we provide a feasible solution to primal problem (31) and show its objective value is equal to $u(\kappa^*)$ (part 1). Then we construct a feasible solution to the dual of problem (31) and show its objective value is also equal to $u(\kappa^*)$ (part 2). Therefore, weak duality of linear programming implies Theorem 5.

Part 1: A Primal Solution.

We first claim that there exists a feasible solution $\boldsymbol{\alpha}^*$ to the following equations.

$$\begin{array}{ll} \alpha_i^* = 0 & \text{for } i \in \Upsilon_1 \\ \alpha_i^* = \pi_{i,n+1} & \text{for } i \in \Upsilon_2 \\ \alpha_i^* \in (0, \pi_{i,n+1}) & \text{for } i \in \Upsilon_3 \\ \sum_{i=1}^n \left(\mu_i + \bar{d}_i - \frac{\alpha_i^*}{\pi_{i,n+1}} (\underline{d}_i + \bar{d}_i) \right) = T. & \end{array} \tag{A.3}$$

Notice that $u(\kappa)$ is a piecewise linear concave function. If $\kappa^* \in (0, \gamma)$, it must be one of break points, i.e. κ^* must be equal to $\pi_{i,n+1}\underline{d}_i/(\underline{d}_i + \bar{d}_i)$ for some i. Consider a sufficiently small $\epsilon > 0$ such that $[\kappa^* - \epsilon, \kappa^* + \epsilon]$ contains exact one break point κ^*. Since $u(\kappa^*) \geq u(\kappa^* - \epsilon)$, we must have

$$\sum_{i=1}^n \mu_i + \sum_{i \in \Upsilon_1 \cup \Upsilon_3} \bar{d}_i - \underline{d}_i \sum_{i \in \Upsilon_2} \geq T.$$

Similarly, since $u(\kappa^*) \geq u(\kappa^* + \epsilon)$, we must have

$$\sum_{i=1}^n \mu_i + \sum_{i \in \Upsilon_1} \bar{d}_i - \underline{d}_i \sum_{i \in \Upsilon_2 \cup \Upsilon_3} \leq T.$$

Then, we define a continuous function

$$e(\boldsymbol{\alpha}) = \sum_{i=1}^n \left[(\mu_i + \bar{d}_i) - \frac{\alpha_i}{\pi_{i,n+1}} (\underline{d}_i + \bar{d}_i). \right]$$

We know that $e(\underline{\boldsymbol{\alpha}}) \geq T$ where $\underline{\alpha}_i = 0$ for $i \in \Upsilon_1 \cup \Upsilon_3$ and $\underline{\alpha}_i = \pi_{i,n+1}$ for $i \in \Upsilon_2$. On the other hand, we have $e(\bar{\boldsymbol{\alpha}}) \leq T$ where $\bar{\alpha}_i = 0$ for $i \in \Upsilon_1$ and $\bar{\alpha}_i = \pi_{i,n+1}$ for $i \in \Upsilon_2 \cup \Upsilon_3$. Thus, there must exist an $\boldsymbol{\alpha}^* \in [\underline{\boldsymbol{\alpha}}, \bar{\boldsymbol{\alpha}}]$ such that $e(\boldsymbol{\alpha}^*) = T$ which implies the existence of a feasible $\boldsymbol{\alpha}^*$ for the system of equations (A.3).

We are now ready to construct the following solution to problem (31).

$$\xi_{ij}^* = \begin{cases} (\pi_{ij} - \alpha_i^*)(\mu_i + \bar{d}_i) & \text{for } \alpha_i^* \leq \pi_{ij} \\ (\pi_{ij} - \alpha_i^*)(\mu_i - \underline{d}_i) & \text{for } \alpha_i^* > \pi_{ij} \end{cases} \tag{A.4}$$

$$\lambda_i^* = \xi_{ii}^*, \qquad \text{for } i = 1, \cdots, n \tag{A.5}$$

$$s_i^* = \mu_i + \bar{d}_i - \frac{\alpha_i^*}{\pi_{i,n+1}} (\underline{d}_i + \bar{d}_i), \qquad \text{for } i = 1, \cdots, n. \tag{A.6}$$

This solution clearly satisfies constraints (32)-(36), and thus is a feasible solution to problem (31). Next, we show that its corresponding objective value $u(\kappa^*)$.

By construction, for $i \in \Upsilon_1$, we have $\alpha_i^* = \lambda_i^* = 0$ and thus $\lambda_i^* + \mu\alpha_i^* = 0$. For $i \in \Upsilon_2$, we have $\alpha_i^* = \pi_{i,n+1}$ and $\lambda_i^* = -\pi_{i,n+1}(\mu_i - \underline{d}_i)$, which implies $\lambda_i^* + \mu_i\alpha_i^* = \pi_{i,n+1}\underline{d}_i$. For $i \in \Upsilon_3$, we have

$$\begin{aligned}
&\sum_{i\in\Upsilon_3}(\lambda_i^* + \mu_i\alpha_i^*)\\
=&\sum_{i\in\Upsilon_3}(-\alpha_i^*(\mu_i - \underline{d}_i) + \mu_i\alpha_i^*)\\
=&\sum_{i\in\Upsilon_3}\underline{d}_i\alpha_i^*\\
=&\,\kappa^*\sum_{i\in\Upsilon_3}(\underline{d}_i + \bar{d}_i)\frac{\alpha_i^*}{\pi_{i,n+1}}
\end{aligned}$$

where the last equality follows from the fact that $\kappa^* = \pi_{i,n+1}\underline{d}_i/(\underline{d}_i + \bar{d}_i)$ for $i \in \Upsilon_3$. However, by (A.3), $\alpha_i^* = 0$ for $i \in \Upsilon_1$, and

$$\sum_{i=1}^{n}\left(\mu_i + \bar{d}_i - \frac{\alpha_i^*}{\pi_{i,n+1}}(\underline{d}_i + \bar{d}_i)\right) = T.$$

It then follows that

$$\kappa^*\sum_{i\in\Upsilon_3}(\underline{d}_i + \bar{d}_i)\frac{\alpha_i^*}{\pi_{i,n+1}} = \kappa^*(-T + \sum_{i=1}^{n}(\mu_i + \bar{d}_i) - \sum_{i\in\Upsilon_2}(\underline{d}_i + \bar{d}_i)).$$

In sum,

$$\begin{aligned}
\sum_{i=1}^{n}(\lambda_i^* + \mu_i\alpha_i^*) &= \sum_{i\in\Upsilon_2}(\lambda_i^* + \mu_i\alpha_i^*) + \sum_{i\in\Upsilon_3}(\lambda_i^* + \mu_i\alpha_i^*)\\
&= \sum_{i\in\Upsilon_2}\pi_{i,n+1}\underline{d}_i + \kappa^*\sum_{i\in\Upsilon_3}(\underline{d}_i + \bar{d}_i)\frac{\alpha_i^*}{\pi_{i,n+1}}\\
&= \sum_{i\in\Upsilon_2}\pi_{i,n+1}\underline{d}_i + \kappa^*\left(-T + \sum_{i=1}^{n}(\mu_i + \bar{d}_i) - \sum_{i\in\Upsilon_2}(\underline{d}_i + \bar{d}_i)\right)\\
&= \left(\sum_{i=1}^{n}\mu_i - T\right)\kappa^* + \sum_{i\in\Upsilon_1\cup\Upsilon_3}\bar{d}_i\kappa^* + \sum_{i\in\Upsilon_2}\underline{d}_i(\pi_{i,n+1} - \kappa^*)\\
&= \left(\sum_{i=1}^{n}\mu_i - T\right)\kappa^* + \sum_{i=1}^{n}\min(\bar{d}_i\kappa^*, \underline{d}_i(\pi_{i,n+1} - \kappa^*))\\
&= u(\kappa^*),
\end{aligned}$$

where the second last equality holds by the definition of Υ_i for $i = 1, 2, 3$.

Part 2: A Dual Solution.

Let δ_{kj} be the dual variable associated with constraint (32) for $1 \le k \le n$ and $k \le j \le n+1$, ϑ_{ij} be the dual variable associated with constraint (33) for $1 \le i \le n$ and $i \le j \le n+1$, ι_{ij} be the dual variable associated with constraint (34) for $1 \le i \le n$ and $i \le j \le n+1$, and κ be the dual variable associated with constraint (35). Then the dual problem of (31) is.

$$\max_{\kappa,\boldsymbol{\delta},\boldsymbol{\vartheta},\boldsymbol{\iota}\ge 0} \sum_{i=1}^{n}\sum_{j=i}^{n+1}\left[\pi_{ij}(\mu_i-\underline{d}_i)\vartheta_{ij}+\pi_{ij}(\mu_i+\bar{d}_i)\iota_{ij}\right]-\kappa T \tag{A.7}$$

$$s.t. \quad \sum_{k=1}^{i}\sum_{j=i}^{n+1}\delta_{kj}=1 \qquad \text{for } 1\le i\le n \tag{A.8}$$

$$\sum_{k=1}^{i}\sum_{j=i}^{n+1}\pi_{ij}\delta_{kj}\le\kappa \qquad \text{for } 1\le i\le n \tag{A.9}$$

$$\sum_{k=1}^{i}\delta_{kj}=\vartheta_{ij}+\iota_{ij} \qquad \text{for } 1\le i\le n, 1\le i\le j\le n+1 \tag{A.10}$$

$$\sum_{j=i}^{n+1}\left[\vartheta_{ij}(\mu_i-\underline{d}_i)+\iota_{ij}(\mu_i+\bar{d}_i)\right]=\mu_i \qquad \text{for } 1\le i\le n. \tag{A.11}$$

We construct a dual solution as follows. Let

$$\begin{aligned}
\delta^*_{1,n+1} &= \tfrac{\kappa^*}{\pi_{1,n+1}} \\
\delta^*_{i,n+1} &= \tfrac{\kappa^*}{\pi_{i,n+1}}-\tfrac{\kappa^*}{\pi_{i-1,n+1}} && \text{for } i=2,\cdots,n \\
\delta^*_{ii} &= 1-\tfrac{\kappa^*}{\pi_{i,n+1}} && \text{for } i=1,\cdots,n \\
\delta^*_{ij} &= 0 && \text{for } i<j\le n.
\end{aligned}$$

For any i with $\bar{d}_i\kappa^*\le\underline{d}_i(\pi_{i,n+1}-\kappa^*)$, we let

$$\begin{aligned}
\iota^*_{i,n+1} &= \tfrac{\kappa^*}{\pi_{i,n+1}} \\
\iota^*_{ii} &= \tfrac{\underline{d}_i}{\bar{d}_i+\underline{d}_i}-\tfrac{\kappa^*}{\pi_{i,n+1}} \\
\iota^*_{ij} &= 0 && \text{for } i<j\le n \\
\vartheta^*_{ii} &= \tfrac{\bar{d}_i}{\bar{d}_i+\underline{d}_i} \\
\vartheta^*_{ij} &= 0 && \text{for } i<j\le n+1.
\end{aligned}$$

For any i with $\bar{d}_i\kappa^*>\underline{d}_i(\pi_{i,n+1}-\kappa^*)$, we let

$$\begin{aligned}
\iota^*_{i,n+1} &= \tfrac{\underline{d}_i}{\bar{d}_i+\underline{d}_i} \\
\iota^*_{ij} &= 0 && \text{for } i\le j\le n \\
\vartheta^*_{i,n+1} &= \tfrac{\kappa^*}{\pi_{i,n+1}}-\tfrac{\underline{d}_i}{\bar{d}_i+\underline{d}_i} \\
\vartheta^*_{ii} &= 1-\tfrac{\kappa^*}{\pi_{i,n+1}} \\
\vartheta^*_{ij} &= 0 && \text{for } i<j\le n.
\end{aligned}$$

It is straightforward to verify that the above solution is feasible to the dual problem because it ensures constraints (A.8)-(A.11) to hold as equality. Moreover, its associated objective value is $\left(\sum_{i=1}^n\mu_i-T\right)\kappa^*+\sum_{i=1}^n\min(\bar{d}_i\kappa^*,\underline{d}_i(\pi_{i,n+1}-\kappa^*))$. This completes the proof. □

Proof of Theorem 6

Since $\pi_{i,n+1}\ge\gamma\ge\kappa_\psi$, $\pi_{i,n+1}-\kappa_\psi$ must be positive and decreasing in i. Following the same proof as that of Theorem 3, we have

$$G(\psi)\ \ge\ \left(\sum_{i=1}^{n}\mu_\psi-T\right)\kappa_\psi+L_\psi\sum_{i=1}^{n}\min(\varphi\kappa_\psi,(1-\varphi)(\pi_{i,n+1}-\kappa_\psi))$$

$$
\begin{aligned}
&\geq \left(\sum_{i=1}^{n} \mu_{\psi} - T\right) \kappa_{\psi^*} + L_{\psi} \sum_{i=1}^{n} \min(\varphi \kappa_{\psi^*}, (1-\varphi)(\pi_{i,n+1} - \kappa_{\psi^*})) \\
&\geq \left(\sum_{i=1}^{n} \mu_{\psi^*} - T\right) \kappa_{\psi^*} + L_{\psi^*} \sum_{i=1}^{n} \min(\varphi \kappa_{\psi^*}, (1-\varphi)(\pi_{i,n+1} - \kappa_{\psi^*})) \\
&= G(\psi^*)
\end{aligned}
$$

This completes the proof. □

Proof of Lemma 8

Proof: We introduce a new variable z to denote $\sum_{j=1}^{m} a_j x_j$. By assumption, for any feasible solution $\boldsymbol{x}$, $\sum_{j=1}^{m} a_j x_j \in [0, a_m]$. Then, problem (39) can be reformulated as

$$\min_{z \in [0, a_m]} \; opt(z), \tag{A.12}$$

where, for any given $z \in [0, a_m]$,

$$
\begin{aligned}
opt(z) = \max\nolimits_{\mathbf{x}} \; & \sqrt{\sum_{j=1}^{m} a_j^2 x_j - z^2} - bz \\
\text{s.t.} \; & \sum_{j=1}^{m} x_j = 1 \\
& \sum_{j=1}^{m} a_j x_j = z \\
& x_j \geq 0, \quad \text{for } j = 1, \cdots, m.
\end{aligned} \tag{A.13}
$$

When z is fixed, the objective function of problem (A.13) is strictly increasing in $\sum_{j=1}^{m} a_j^2 x_j$. Thus, any optimal solution to problem (A.13) is also optimal to the following problem

$$
\begin{aligned}
\max\nolimits_{\mathbf{x}} \; & \sum_{j=1}^{m} a_j^2 x_j \\
\text{s.t.} \; & \sum_{j=1}^{m} x_j = 1 \\
& \sum_{j=1}^{m} a_j x_j = z \\
& x_j \geq 0, \quad \text{for } j = 1, \cdots, m,
\end{aligned} \tag{A.14}
$$

and vice versa.

We now solve problem (A.14) for any given $z \in [0, a_m]$. The problem is a linear program with two linear constraints, besides the nonnegativity constraints. Thus, there exists an optimal solution, denoted by $\boldsymbol{x}(z)$, which has at most two non-zero variables. Then suppose that the two non-zero variables are $x_i(z) > 0$ and $x_k(z) \geq 0$. And $x_j(z) = 0$ for all $j \neq i, k$. Without loss of generality, we assume that $i \leq k$. From the constraints of problem (A.14), we must have $x_k(z) = 1 - x_i(z)$, and

$$z = a_i x_i(z) + a_k x_k(z) = a_i x_i(z) + a_k(1 - x_i(z)) = a_k - (a_k - a_i) x_i(z).$$

It follows that

$$x_i(z) = \frac{a_k - z}{a_k - a_i} > 0, \quad x_k(z) = \frac{z - a_i}{a_k - a_i} \geq 0.$$

Therefore, the optimal objective value of (A.14) is given by

$$a_i^2 x_i(z) + a_k^2 x_k(z) = a_k z + a_i(z - a_k) \leq a_k z \leq a_m z$$

where the first inequality holds because $z - a_k \leq 0$ and the second holds because $a_k \leq a_m$ and $z \geq 0$. That is, the optimal objective value of (A.14) is bounded above by $a_m z$, which is attainable when $i = 1$ and $k = m$. This shows that $x_1(z) = 1 - \frac{z}{a_m}$, $x_m(z) = \frac{z}{a_m}$. Therefore, for any given $z \in [0, a_m]$,

$$opt(z) = \sqrt{a_m z - z^2} - bz.$$

By Lemma 7, $z^* = \frac{a_m}{2}\left[1 - \frac{b}{\sqrt{1+b^2}}\right]$ maximizes $opt(z)$ in $[0, a_m]$. And $(\boldsymbol{x}(z^*), z^*)$ is an optimal solution to problem (39). The lemma follows by noticing that $x_1(z^*) = \frac{1}{2} + \frac{b}{2\sqrt{1+b^2}}$, $x_m(z^*) = \frac{1}{2} - \frac{b}{2\sqrt{1+b^2}}$, and $x_j(z^*) = 0$ for $1 < j < m$. $\square$